Current in a two-dimensional ratchet with an asymmetric unbiased external force

2007 J. Phys. A: Math. Theor. 40 6061

(http://iopscience.iop.org/1751-8121/40/23/004)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 6061–6070 doi:10.1088/1751-8113/40/23/004

Current in a two-dimensional ratchet with an asymmetric unbiased external force

Bao-quan Ai[1], **Feng-guo Li**[1] **and Liang-gang Liu**[2]

[1] Institute for Condensed Matter Physics, School of Physics and Telecommunication Engineering and Laboratory of Photonic Information Technology, South China Normal University, 510006 GuangZhou, People's Republic of China
[2] Faculty of Science and Technology, Macau University of Science and Technology, Macao

E-mail: aibq@scnu.edu.cn

Received 10 January 2007, in final form 12 April 2007
Published 22 May 2007
Online at stacks.iop.org/JPhysA/40/6061

Abstract

We study the transport of an overdamped Brownian particle moving in an asymmetric two-dimensional periodic potential in the presence of an asymmetric unbiased external force. Based on the effective potential approach, the two-dimensional system is simplified to the one-dimensional system. The expression of the net current is obtained at a quasi-steady-state limit. The competitions among the asymmetric parameter between the potentials, the coupling parameter of the potential and the temporally asymmetric parameter of the external force lead to current reversals. There may be three optimized values of temperature at which the current takes its extremum value.

PACS numbers: 05.60.Cd, 05.40.Jc, 02.50.Ey

1. Introduction

Recently, there has been increasing interest in studying the noise-induced transport of Brownian particles for systems with a spatially periodic potential field [1]. This comes from the desire of understanding molecular motors [2], nanoscale friction [3], surface smoothening [4], coupled Josephson junctions [5], optical ratchets and directed motion of laser cooled atoms [6] and mass separation and trapping schemes at microscale [7].

The focus of research has been on the noise-induced unidirectional motion over the last decade. A ratchet system is generally defined as a system that is able to transport particles in a periodic structure with nonzero macroscopic velocity in the absence of macroscopic force on average. In these systems, directed Brownian motion of particles is generated by nonequilibrium noise in the absence of any net macroscopic forces and potential gradients [1]. Typical examples are rocking ratchets [1, 8], flashing ratchets [9], diffusion ratchets [10] and correlation ratchets [1, 11]. In all these studies, the potential is taken to be asymmetric in space. It has also been shown that a unidirectional current can also appear for spatially symmetric potentials if there exits an external random force either asymmetric or spatially dependent [1].

The study of current reversal phenomena has given rise to much research activity on its own; the motivation being the possibility of new particle separation devices superior to existing methods such as the electrophoretic method for particles of micrometre scale [12]. It is known that current reversals in ratchet systems can be engendered by varying the system parameters. The current can be reversed, for example, by a noise of Gaussian force with non-white power spectrum in the presence of the stationary periodic potential [13]. The current reversal can also be obtained in two-state ratchets if the long arm is kinked [14]. Bier and Astumian [15] have also found the current reversal in a fluctuating three-state ratchet. In the presence of a kangaroo process as the driving force, the current reversal can be triggered by varying the noise flatness, the ratio of the fourth moment to the square of the second moment [16]. The current reversal can be induced by both an additive Gaussian white and an additive Ornstein–Uhlenbeck noise in a correlation ratchet [17]. The current reversal also appears in forced inhomogeneous ratchets [18]. In the temperature ratchet the competition between the asymmetric parameter of the potential and the temperature difference may lead to current reversals [19]. There exist already several works on current reversals in higher dimensional systems [20]. The current reversal was also obtained in the case of a chemical system [21].

The previous works on the current reversal are limited to the case of one-dimensional systems and symmetric temporal driving forces. The present work extends the study of the current reversal to the case of the three competitive driving factors: the asymmetry of the potential, the coupling between the potentials and the asymmetry of the external force. Based on the effective potential approach, the 2D problem is simplified to the 1D problem with a reduced effective potential through eliminating the variable y. It is found that when a driving factor competes with another one, the current may reverse its direction. There exist two or three values of temperature at which the current takes its extremum value. Our emphasis is on finding current reversals and the optimized values of temperature. This is achieved by using a quasi-steady-state limit to solve the Fokker–Planck equation.

2. Average current of the two-dimensional ratchet

In this paper, we consider the movement of an overdamped Brownian particle moving in a two-dimensional coupled asymmetric periodic potential with a temporally asymmetric unbiased external force. The ratchet model is described by the following 2D Langevin equation written in a dimensionless form [22],

$$\begin{aligned} \dot{x} &= -\frac{\partial U(x,y)}{\partial x} + F(t) + \sqrt{2k_B T}\xi_x(t), \\ \dot{y} &= -\frac{\partial U(x,y)}{\partial y} + \sqrt{2k_B T}\xi_y(t), \end{aligned} \tag{1}$$

where x, y stands for the position of the Brownian particle, k_B is the Boltzmann constant and T is the absolute temperature. $\xi_x(t)$ and $\xi_y(t)$ are the randomly fluctuating Gaussian white noises with zero mean and with the autocorrelation function $\langle \xi_x(t)\xi_y(s)\rangle = \delta_{x,y}\delta(t-s)$. Here $\langle \cdots \rangle$ denotes an ensemble average over the distribution of the fluctuating forces $\xi_{x,y}(t)$. $F(t)$ is a temporally asymmetric unbiased external periodic force (figure 1), satisfying [23, 24]

$$F(t+\tau) = F(t), \qquad \int_0^\tau F(t)\,\mathrm{d}t = 0, \tag{2}$$

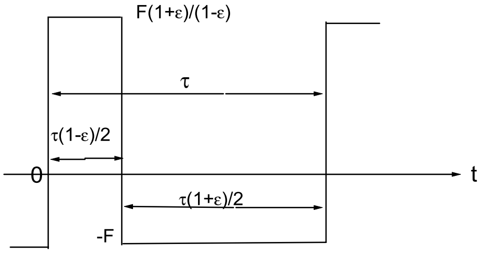

Figure 1. Driving force: $F(t)$ which preserved the zero mean $\langle F(t)\rangle = 0$; $F(t+\tau) = F(t)$; ε is the temporally asymmetry parameter.

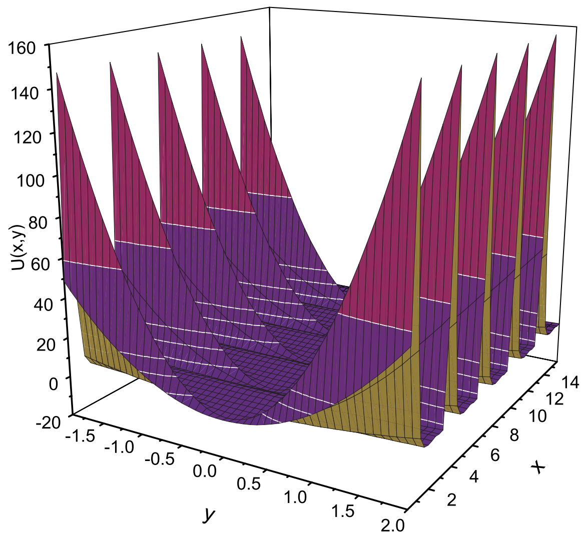

Figure 2. The two-dimensional ratchet potential (equation (5)) with $U_0 = 2$, $\Delta = 1.0$ and $\lambda = -2$.

$$F(t) = \begin{cases} \dfrac{1+\varepsilon}{1-\varepsilon} F_0, & n\tau \leqslant t < n\tau + \dfrac{1}{2}\tau(1-\varepsilon), \\ -F_0, & n\tau + \dfrac{1}{2}\tau(1-\varepsilon) < t \leqslant (n+1)\tau, \end{cases} \tag{3}$$

where τ is the period of the driving force, F_0 its magnitude and ε the temporally asymmetric parameter.

The potential $U(x, y)$ is periodic along the x-direction and parabolic along the y-direction,

$$U(x, y) = U_1(x) + \tfrac{1}{2}C(x)y^2, \tag{4}$$

with

$$U_1(x) = -U_0\left[\sin(x) + \tfrac{1}{4}\Delta\sin(2x)\right], \qquad C(x) = \exp[\lambda U_1(x)], \tag{5}$$

where $U_1(x)$ is the potential along the x-direction, U_0 is the height of the barrier and Δ is the asymmetric parameter of the potential along the x-direction. The periodic length of $C(x)$ is equal to that of $U_1(x)$. λ is the coupling parameter between the potentials along x and y. The form of $C(x)$ is chosen to be equation (5) in order to observe a strong coupling between x and y degrees of freedom. For any given x, a slice of the potential along the direction of y is a parabola, and this parabola becomes wide and narrow, periodically. $U(x, y)$ picture with $U_0 = 2$, $\Delta = 1.0$ and $\lambda = -2$ is plotted in figure 2.

Based on the effective potential approach [22, 25], the 2D problem is simplified to the 1D problem with a reduced potential $U_{\rm eff}(x)$ through eliminating the variable y. We can integrate over y from $-\infty$ to ∞ in the equilibrium distribution, thus the effective potential is given by

$$U_{\rm eff}(x) = -k_B T \ln\left\{\int_{-\infty}^{\infty} {\rm d}y \exp\left[-\frac{U(x,y)}{k_B T}\right]\right\},$$
$$= U_1(x) + \frac{1}{2}k_B T \ln\left[\frac{C(x)}{2\pi k_B T}\right]. \tag{6}$$

Hence, equation (1) reduces to

$$\dot{x} = -\frac{\partial U_{\rm eff}(x)}{\partial x} + F(t) + \sqrt{2k_B T}\xi_x(t). \tag{7}$$

The evolution of the probability density for x is given by the associated Fokker–Planck equation [1, 26]

$$\frac{\partial P(x,t)}{\partial t} = \frac{\partial}{\partial x}\left\{k_B T\frac{\partial P(x,t)}{\partial x} + [U'_{\rm eff}(x) - F(t)]P(x,t)\right\} = -\frac{\partial j(x,t)}{\partial x}, \tag{8}$$

$$j(x,t) = -[U'_{\rm eff}(x) - F(t)]P(x,t) - \frac{\rm d}{{\rm d}x}[k_B T P(x,t)], \tag{9}$$

here $j(x,t)$ is the probability current density. The prime stands for the derivative with respect to the space variable x. $P(x,t)$ is the probability density for the particle at position x and time t. It satisfies the normalization condition and the periodicity condition:

$$P(x,t) = P(x+2\pi,t), \tag{10}$$

$$\int_0^{2\pi} P(x,t)\,{\rm d}x = 1. \tag{11}$$

If $F(t)$ changes very slowly with respect to t, namely, its period is longer than any other time scale of the system, there exists a quasi-steady state. In this case, $j(x,t)$ is a constant. From equation (10) we can obtain the solution of equation (9),

$$P(x,t) = \frac{j}{k_B T\left\{1-\exp\left[\frac{-2\pi F(t)}{k_B T}\right]\right\}}\exp\left[-\frac{U_{\rm eff}(x)-F(t)x}{k_B T}\right]\int_x^{x+2\pi}\exp\left[\frac{U_{\rm eff}(y)-F(t)y}{k_B T}\right]{\rm d}y. \tag{12}$$

Substituting equation (12) into equation (11), we can obtain the current

$$j(F(t)) = \frac{k_B T\left\{1-\exp\left[-\frac{2\pi F(t)}{k_B T}\right]\right\}}{\int_0^{2\pi}\exp\left[-\frac{U_{\rm eff}(x)-F(t)x}{k_B T}\right]{\rm d}x\int_x^{x+2\pi}\exp\left[\frac{U_{\rm eff}(y)-F(t)y}{k_B T}\right]{\rm d}y}. \tag{13}$$

The average current is

$$J = \frac{1}{\tau}\int_0^{\tau} j(F(t))\,{\rm d}t. \tag{14}$$

For a temporally asymmetric force $F(t)$, we can obtain

$$J = \tfrac{1}{2}(j_1 + j_2) \tag{15}$$

with

$$j_1 = (1-\varepsilon)j\left(\frac{1+\varepsilon}{1-\varepsilon}F_0\right), \qquad j_2 = (1+\varepsilon)j(-F_0). \tag{16}$$

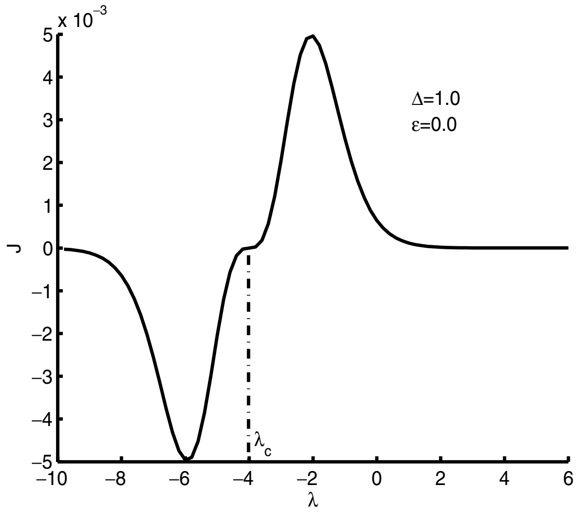

Figure 3. Current J versus the coupling parameter λ between the potentials at $U_0 = 2.0$, $F_0 = 0.5$, $\lambda_c = -4.0$, $T = 0.5$, $\Delta = 1.0$ and $\varepsilon = 0.0$.

3. Results and discussions

Figure 3 shows the current J as a function of the coupling parameter λ between the potentials at $\Delta = 1.0$ and $\varepsilon = 0.0$. From equation (6), we can obtain

$$U_{\text{eff}}(x) = \left(1 + \tfrac{1}{2}\lambda k_B T\right)U_1(x) - \tfrac{1}{2}k_B T \ln 2\pi k_B T. \tag{17}$$

When $1 + \frac{1}{2}\lambda k_B T = 0$, namely $\lambda = \lambda_c = -4.0$ at $k_B T = 0.5$, the effective potential reduces to a constant potential and the ratchet effect disappears, so the current is zero. When $\lambda < \lambda_c$, the current J is negative. It is easy to find that when $\lambda \ll \lambda_c$, the height of the potential is very high and the particle cannot pass the barrier, so no current can be obtained. There exists a value of λ at which the current J takes its minimum value. Similarly, when $\lambda > \lambda_c$, the current is positive and has a maximum value. Therefore, we can obtain the current reversal by changing the value of λ, the coupling parameter between the potentials.

Figure 4(*a*) shows the current J as a function of the negative temporally asymmetric parameter ε of the driving force at $\Delta = 0$ and $\lambda = 0$. When $\varepsilon = -1$, namely, no external force acting on the system, J is equal to $j(0) = 0$, so that no current occurs. When $\varepsilon = 0$, namely, the driving force is temporally symmetric, therefore, no current can be obtained for a symmetric potential, also. When $-1 < \varepsilon < 0$, the current J is negative and has a minimum value. The current J versus the positive temporally asymmetric parameter ε is shown in figure 4(*b*). The current J is positive for $\varepsilon > 0$ and increases with ε. When we change the sign of ε, the current reversal can occur. The current J has the nonlinear function of ε. The temporally asymmetry is a way of inducing a net current.

The current J versus the asymmetric parameter Δ of the potential is shown in figure 5 at $\lambda = 0.0$ and $\varepsilon = 0.0$. When $\Delta = 0.0$, namely, the potential is symmetric, the current vanishes for the symmetric external force. When $\Delta < 0$, the current is negative. When $\Delta \ll 0$, the height of the potential is very high and the particle cannot pass the barrier, so there is no current. There is a value of Δ at which the current J takes its minimum value. Similarly, when $\Delta > 0$, the current is positive and there exists a maximum value. Hence, the current can

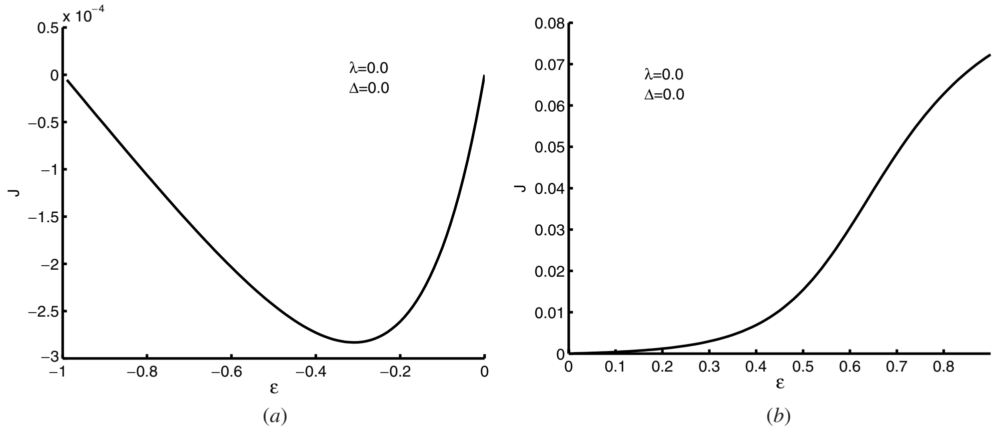

Figure 4. Current J versus the temporally asymmetric parameter ε of the driving force at $U_0 = 2.0$, $F_0 = 0.5$, $T = 0.5$, $\Delta = 0.0$ and $\lambda = 0.0$: (*a*) ε is negative; (*b*) ε is positive.

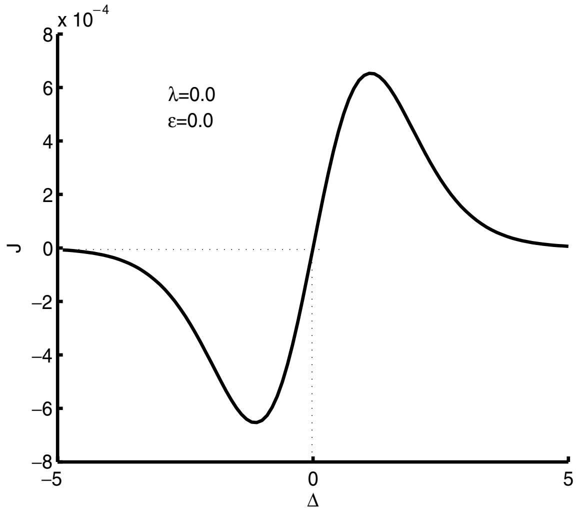

Figure 5. Current J versus the asymmetric parameter Δ between the potentials at $U_0 = 2.0$, $F_0 = 0.5$, $T = 0.5$, $\lambda = 0.0$ and $\varepsilon = 0.0$.

reverse its direction by changing the sign of Δ. The asymmetry of the potential is a factor for obtaining a net current.

In figure 6, we plot the current J as a function of temperature T for different values of the coupling parameter λ at $\Delta = 1.0$ and $\varepsilon = 0.0$. In this case, the competition between the asymmetry parameter Δ of the potential and the coupling parameters λ is investigated. When $T \to 0$, J tends to zero. Therefore, the particle cannot pass the barrier and there is no current. When $T \to \infty$, so that the thermal noise is very large, the ratchet effect disappears and J tends to zero, also. When $\lambda = -1.0$, the current is positive, there exists a peak. As the coupling parameter λ decreases, the position of the peak moves to zero. A vale even appears for $\lambda = -5.0$. The current may reverse its direction as increasing temperature T. The competition between Δ and λ may lead to current reversal.

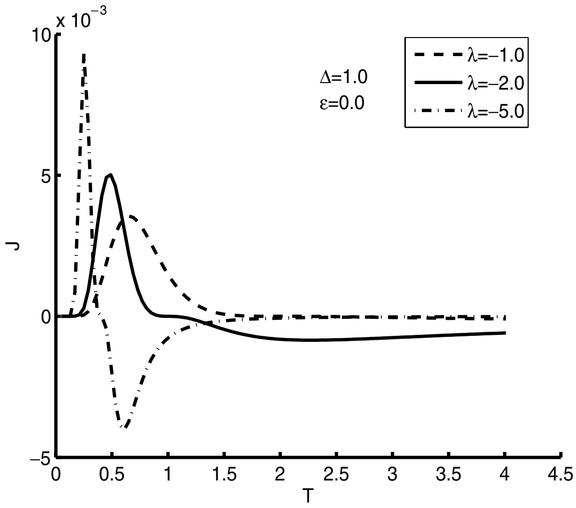

Figure 6. Current J versus temperature T for different values of the coupling parameters λ at $U_0 = 2.0$, $F_0 = 0.5$, $\Delta = 1.0$ and $\varepsilon = 0.0$.

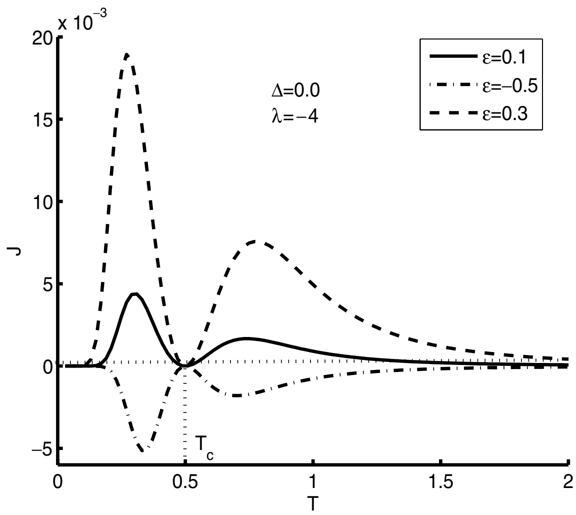

Figure 7. Current J versus temperature T for different values of the asymmetric parameters of the driving force ε at $U_0 = 2.0$, $F_0 = 0.5$, $T_c = 0.5$, $\Delta = 0.0$ and $\lambda = -4.0$.

Figure 7 shows the current J as a function of temperature T for different values of the asymmetric parameters ε of the driving force at $\Delta = 0.0$ and $\lambda = -4.0$. In this case, we discuss the competition between λ and ε. From equation (17), when $T = T_c = 0.5$, the potential reduces to a constant and no current can be obtained. When $T \to 0$ and $T \to \infty$, the current J tends to zero, also. So there are two optimized values of T at which the current J takes its extremum values. There exist two peaks or two vales in J–T figure.

The current J as a function of temperature T is shown in figure 8 for the combination of $\Delta = 1.0, \lambda = -6.0$ and $\varepsilon = 0.2$. In this case, the competitions among the asymmetric parameter of the potential, the coupling parameter of the potential and the temporally

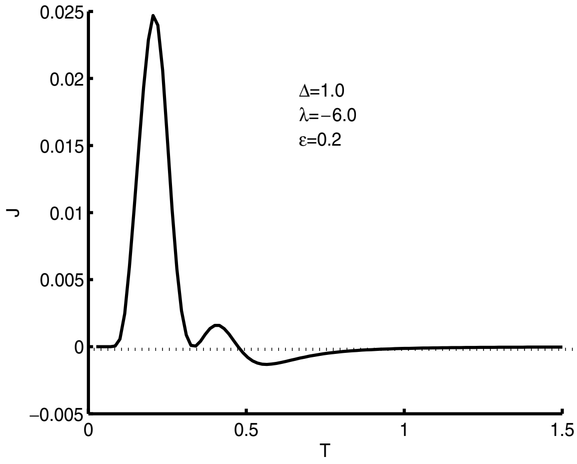

Figure 8. Current J versus temperature T at $U_0 = 2.0$, $F_0 = 0.5$, $\Delta = 1.0$, $\varepsilon = 0.2$ and $\lambda = -6.0$.

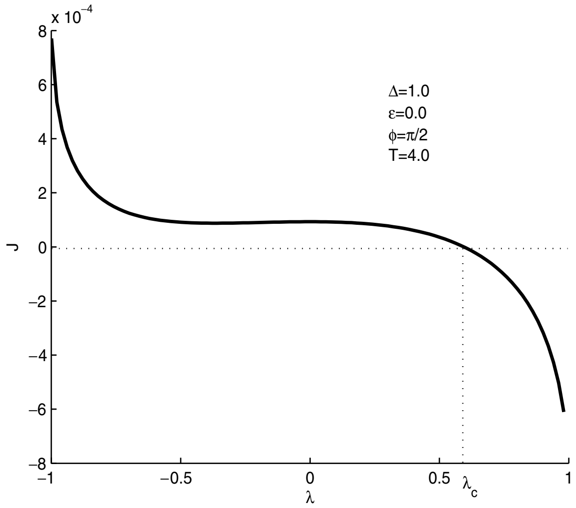

Figure 9. Current J versus the coupling parameter λ between the potentials for the case of $C(x) = 1 - \lambda\sin(x+\phi)$. $\phi = \pi/2$, $T = 4.0$ and the other parameters are the same as that in figure 3.

asymmetric parameter of the external force are investigated. From the figure, it is easy to find that there are three values of T at which the current J takes its extremum value. There are two peaks and a vale in figure 8. The current can reverse its direction as the temperature T changes.

The form of $C(x)$ is chosen to be equation (5) in order to observe a strong coupling between x and y degrees of freedom. This coupling term is a very special form which makes the analytic calculations. In order to prove the generality of our findings, we investigate the system with a different coupling term,

$$C(x) = 1 - \lambda\sin(x+\phi), \tag{18}$$

where $0 \leqslant |\lambda| < 1$ and ϕ is a phase shift. The numerical results are shown in figures 9 and 10.

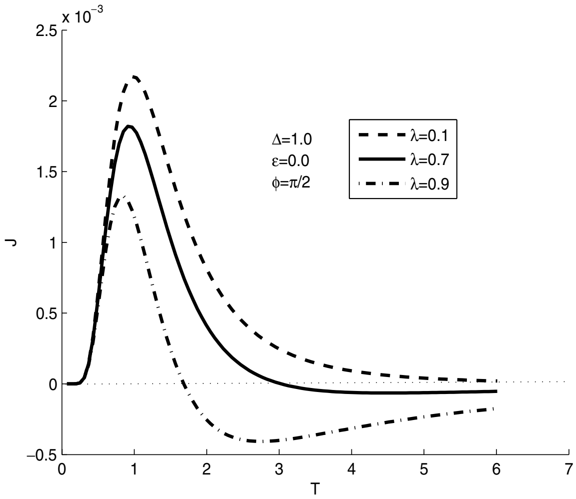

Figure 10. Current J versus temperature T for different values of the coupling parameters λ for the case of $C(x) = 1 - \lambda\sin(x + \phi)$. $\phi = \pi/2$ and the parameters are the same as that in figure 6.

Figure 9 shows the current J as a function of λ for the coupling term given by equation (18). It is found that the current is positive for $\lambda > \lambda_c$, zero at $\lambda = \lambda_c$ and negative for $\lambda < \lambda_c$. Therefore, we can obtain the current reversal by changing the value of the coupling parameter. The result is similar to that in figure 3.

Figure 10 shows the current as a function of temperature T for different values of the coupling parameters with the coupling term given by equation (18). In this case, we investigate the competition between the asymmetry parameter Δ of the potential and the coupling parameter λ. When $\lambda = 0.1$, the current is always positive and there exists a peak. As the coupling parameter λ increases, the height of the peak decreases and a vale appears at $\lambda = 0.9$. For $\lambda = 0.7$ and 0.9, the current may reverse its direction as increasing temperature T. The competition between Δ and λ may lead to current reversals. The same results can be found in figure 6.

In order to prove the generality of the previous results, we have numerically studied a different coupling term (see equation (18)) finding that our conclusions are indeed similar to those analytically obtained for the choice made in equation (5).

4. Concluding remarks

We study the transport of an overdamped Brownian particle moving in a two-dimensional asymmetric periodic potential in the presence of an asymmetric unbiased external force. Based on the effective potential approach, the 2D problem is simplified to the 1D problem with a reduced effective potential through eliminating the variable y. It is found that the asymmetric parameter ε of the external force, the asymmetric parameter Δ of the potential and the coupling parameter between the potentials are the three pivotal factors for obtaining a net current. When two or more driving factors compete each other, the current may reverse its direction. When the three driving factors act on the system, there may be two or three optimized values of temperature T at which current J takes its extremum value. As the traditional viewpoint thinks there is only one peak in J–T figure, there may exist two peaks and one vale in our coupling system. We also investigate the transport for a different coupling term shown in equation (18) and find that the results are similar for the two cases.

Acknowledgments

The work is supported by the National Natural Science Foundation of China under grant no 30600122 and Guangdong Provincial Natural Science Foundation under grant no 06025073.

References

[1] Hanggi P, Marchesoni F and Nori F 2005 *Ann. Phys., Lpz.* **14** 51–70
Reimann P 2002 *Phys. Rep.* **361** 57–265
[2] Maddox J 1994 *Nature (London)* **368** 287
Astumian R D and Derenyi I 1998 *Eur. Biophys. J.* **27** 474
Thomas N and Thornhill R A 1998 *J. Phys. D: Appl. Phys.* **31** 253
Doering C, Ermentrout B and Oster G 1995 *Biophys. J.* **69** 2256
[3] Krim J, Solina D H and Chiarello R 1991 *Phys. Rev. Lett.* **66** 181
Sokoloff J B, Krim J and Widom A 1993 *Phys. Rev.* B **48** 9134
Daikhin L and Urbakh M 1994 *Phys. Rev.* E **49** 1424
Daly C and Krim J 1996 *Phys. Rev. Lett.* **76** 803
Sorensen M R, Jacobsen K W and Stoltze P 1996 *Phys. Rev.* B **53** 2101
[4] Derenyi I, Lee C and Barabasi A-L 1998 *Phys. Rev. Lett.* **80** 1473
[5] Zapata I, Bartussek R, Sols F and Hanggi P 1996 *Phys. Rev. Lett.* **77** 2292
[6] Faucheux L P, Bourdieu L S, Kaplan P D and Libchaber A J 1995 *Phys. Rev. Lett.* **74** 1504
Mennerat-Robilliard C, Lucas D, Guibal S, Tabosa J, Jurczak C, Courtois J-Y and Grynberg G 1999 *Phys. Rev. Lett.* **82** 851
[7] Gorre-Talini L, Spatz J P and Silberzan P 1998 *Chaos* **8** 650
Derenyi I and Astumian R D 1998 *Phys. Rev.* E **58** 7781
Ertas D 1998 *Phys. Rev. Lett.* **80** 1548
Duke T A J and Austin R H 1998 *Phys. Rev. Lett.* **80** 1552
[8] Magnasco M O 1993 *Phys. Rev. Lett.* **71** 1477
Bartussek R, Hanggi P and Kissner J G 1994 *Europhys. Lett.* **28** 459
[9] Hanggi P and Bartussek R 1996 *Nonlinear Physics of Complex System—Current status and Future Trends vol 476* (Berlin: Springer) p 294
[10] Reimann P, Bartussek R, Haussler R and Hanggi P 1996 *Phys. Lett.* A **215** 26
[11] Doering C R, Horsthemke W and Riordan J 1994 *Phys. Rev. Lett.* **72** 2984
[12] Kettner C, Reimann P, Hanggi P and Muller F 2000 *Phys. Rev.* E **61** 312
[13] Millonas M M and Dykman M I 1994 *Phys. Lett.* A **185** 65
[14] Chauwin J-F, Ajdari A and Prost J 1995 *Europhys. Lett.* **32** 373
[15] Bier M and Astumian R D 1996 *Phys. Rev. Lett.* **76** 4277
[16] Doering C R, Horsthemke W and Riordan J 1994 *Phys. Rev. Lett.* **72** 2984
[17] Bartussek R, Reimann P and Hanggi P 1996 *Phys. Rev. Lett.* **76** 1166
[18] Ai B Q, Wang X J, Liu G T, Xie H Z, Wen D H, Chen W and Liu L G 2004 *Eur. Phys. J.* B **37** 523
Dan D, Mahato M C and Jayannavar A M 2001 *Phys. Rev.* E **63** 056307
[19] Ai B Q, Wang L Q and Liu L G 2005 *Phys. Rev.* E **72** 031101
[20] Jung P, Kissner J G and Hanggi P 1996 *Phys. Rev. Lett.* **76** 3436
Lindner B, Schimansky-Geier L, Reimann P, Hanggi P and Nagaoka M 1999 *Phys. Rev.* E **59** 1417
Sengupta S, Guantes R, Miret-Arts S and Hanggi P 2004 *Physica* A **338** 406
[21] Jullien L and Lemarchand A 2001 *J. Phys. Chem.* B **105** 4415
[22] Bao J D 2001 *Phys. Rev.* E **63** 061112
Bao J D and Zhuo Y Z 1998 *Phys. Lett.* A **239** 228
[23] Chialvo D R and Millonas M M 1995 *Phys. Lett.* A **209** 26
[24] Ai B Q, Xie H Z and Liu L G 2005 *Eur. Phys. J.* B **47** 109
[25] Caratti G, Ferrando R, Spadacini R and Tommei G E 1996 *Phys. Rev.* E **54** 4708
Caratti G, Ferrando R, Spadacini R and Tommei G E 1998 *Chem. Phys.* **235** 157
[26] Risken H 1984 *The Fokker–Planck Equation* (Berlin: Springer)